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A robust and fast algorithm for IMRT optimization with dose-volume constraints

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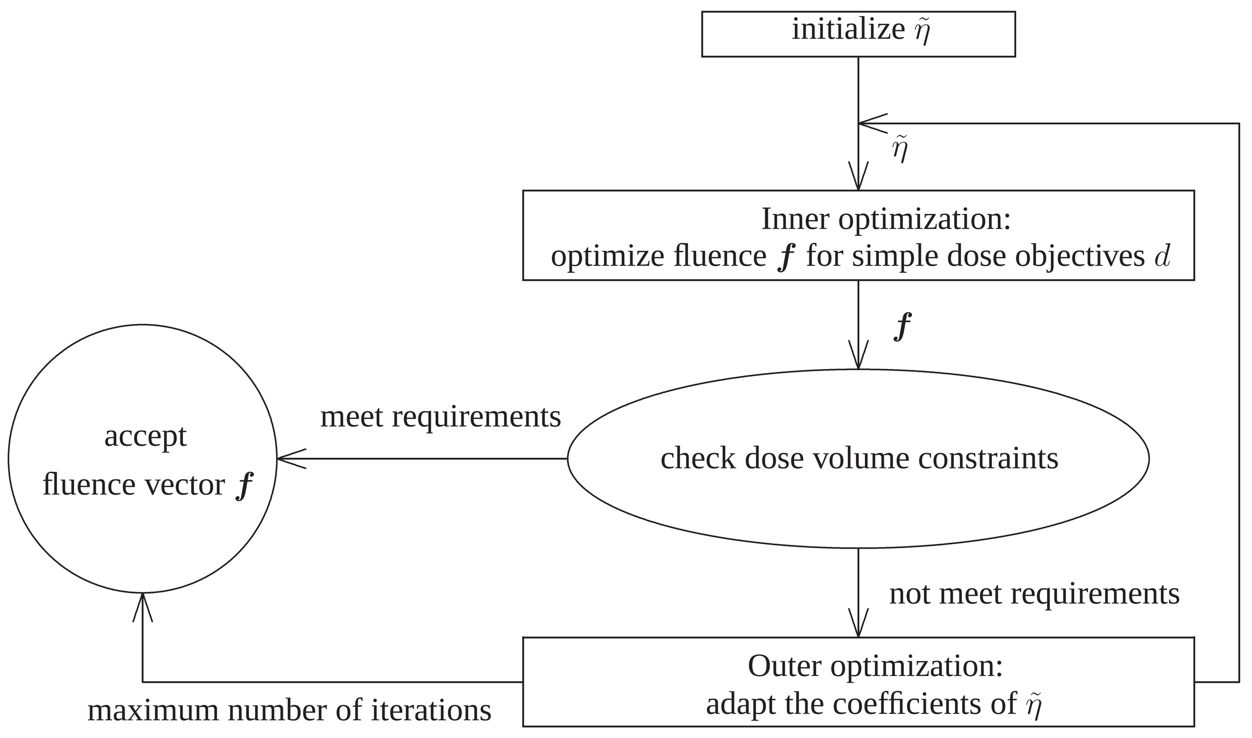
Introduction

The optimization problem in IMRT can usually be described as finding the fluence vector $\boldsymbol{f} \geq 0$ that deliver the dose vector $\boldsymbol{d} = H\boldsymbol{f}$ as close as possible to the prescribed dose $\boldsymbol{d}^p$ in the PTV, while fulfilling as much as possible imposed dose-volume and dose-maximum constraints. In this problem the number and positions of the photon beams are fixed a priori. Each beam is divided in a number of adjacent beamlets; $\boldsymbol{f}$ is a vector which elements are the intensity of these beamlets. $\boldsymbol{d}$ is a vector which elements are the resulting dose in each voxel of the phantom. H is a matrix composed by the dose distribution vectors of the beamlets. The figure below offers the flow chart overall optimization process:



In the inner optimization the fluence vector $\boldsymbol{f}$ is computed that minimizes a quadratic objective function $s(\boldsymbol{f})$ of simple dose objectives $\boldsymbol{d}_v^p$ for the defined structures v. In the outer optimization the voxel dependent factors $\eta_{i,v}$ are adapted, in a way that the voxels i participating the most to the violation of some dose-volume constraints get more importance in the inner optimization.

Inner optimization

The inner optimization uses the following quadratic objective function:

$$s(\boldsymbol{f}) = \sum_v \xi_v (H\boldsymbol{f} - \boldsymbol{d}_v^p)^T \tilde{\eta}_v (H\boldsymbol{f} - \boldsymbol{d}_v^p) + \kappa (M\boldsymbol{f})^T (M\boldsymbol{f}). \tag{1}$$

The first term is the dose objective function where $\boldsymbol{d}_v^p$ are vectors which elements equal the dose objective d_v^p for the voxels in volume v. d_v^p equals the prescribed dose if the structure v is a target volume and 0 if structure v is a risk organ.

The ξ_v are predefined structure-wide importance factors associated with each structure v. The $\tilde{\eta}_v$ are diagonal matrices. The diagonal elements $\eta_{i,v}$ of $\tilde{\eta}_v$ are the voxel importance factors associated to each voxel i of each structure v.

The second term is a smoothing term which works in two ways to the solution of the problem: as a smoothing filter to avoid unachievable frequencies in the fluence, and as a small additional term which facilitates the calculation of the optimal solution of the problem. M is the smoothing matrix and κ a scalar smoothing factor.

Equation 1 can written in a more compact form:

$$s(\boldsymbol{f}) = \frac{1}{2}\boldsymbol{f}^T A \boldsymbol{f} + \boldsymbol{f}^T \boldsymbol{b} + c \tag{2}$$

where:

$$\begin{aligned} A &= H^T Q H + \kappa S \\ \boldsymbol{b} &= H^T \boldsymbol{q} \\ c &= \sum_v \xi_v (\boldsymbol{d}_v^p)^T \tilde{\eta}_v \boldsymbol{d}_v^p \end{aligned} \tag{3}$$

Smoothing term

For the smoothing term we use the discrete Laplace operator Δ on the two dimensional fluence (i.e. the second derivative):

$$\Delta \boldsymbol{f} = M\boldsymbol{f} \tag{4}$$

The first goal of the smoothing term is to get a solution vector $\boldsymbol{f}$ that represent a fluence that is smooth enough to be realized in the pratice.

The second goal is to improve the condition of the matrix A. Without the smoothing term the condition of matrix A varies usually between 10^{15} and infinity. With the smoothing term and $\kappa = 10^{-2} \max \xi_v$ it is usually of the order of 10^5. This improvement in the condition of A is in general an advantage for the calculation time and the precision of the solution. In our case it is of essential importance because it allows to use an algorithm that is faster and more accurate than the gradient methods.

Minimization of the objective function

The matrix A will now be written as:

$$A = H^T Q H + \kappa S \tag{5}$$

where:

$$\begin{aligned} Q &= 2\sum_v \xi_v \tilde{\eta}_v \\ S &= 2M^T M \end{aligned} \tag{6}$$

Q is the only matrix which changes at each iteration of the outer optimization, thus it is important to optimize the calculation of the product $H^T Q H$. An algorithm has been designed which takes advantage that the matrix H is sparse (usually filled for 5 to 20% with nonzero elements) and that Q is a diagonal matrix. In the calculation of the product $H^T Q$, the multiplication with Q is performed while transponing H. The result is then multiplied by H. This approach has two advantages: it is not necessary to store H^T, which is very large, for speeding-up the calculation, and the matrix $H^T Q$ is more sparse than H (since Q is usually sparse too) [1,2].

Another part in the speed improvement is achieved by not using a gradient method but a new algorithm called BOXCQP [3]. BOXCQP solves convex quadratic problems with simple bounds, e.g. $\boldsymbol{f} \geq 0$. It claims to be up to 30 times faster than other algorithms for quadratic programming. Another advantage is that BOXCQP finds the exact minimum and does not need any termination criterium, and it is also simple to implement.

Outer optimization

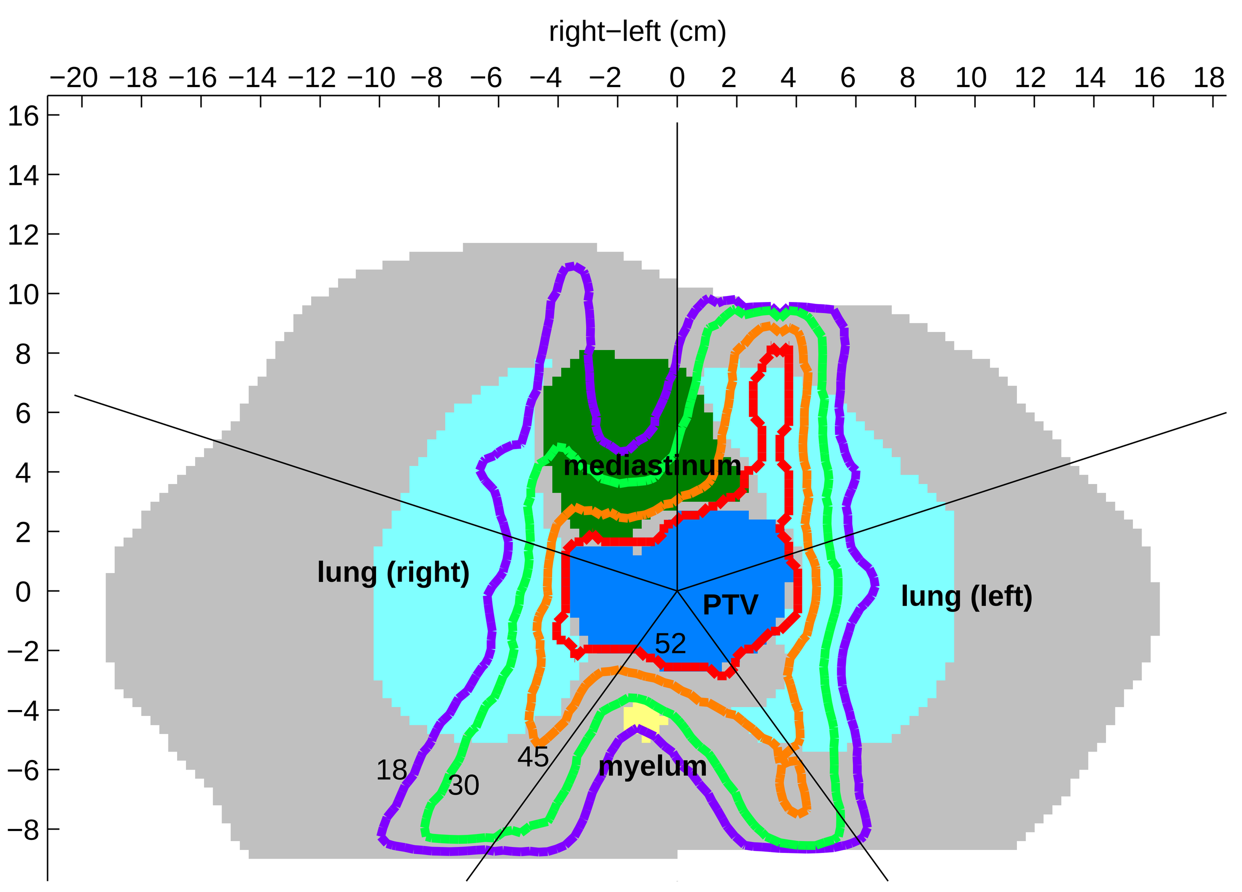
The outer optimization is used to meet the objectives in term of coverage of the PTV, DVHs and maximum doses. For the PTV 100% of the volume must receive a dose higher than 95% of the prescribed dose, and the dose must not exceed a given maximum value. For the organ at risk their dose volume contraint must not be exceeded, i.e. $V(d, d_v^c) < V_v^c$, and the dose must not exceed given maximal values. The idea is to increase the voxel coefficient $\eta_{i,v}$ of the voxels i that participate to the violation of the contraints for structure v. There are several strategies possible; until now the most simple one has given the best results. This simple strategy is to increase after each inner optimization the $\eta_{i,v}$ of the voxels which dose exceed d_v^c in the of dose-volume or dose maximum constraints in the organ at risk, and which dose exceed 95% of d^p in the PTV.

Results

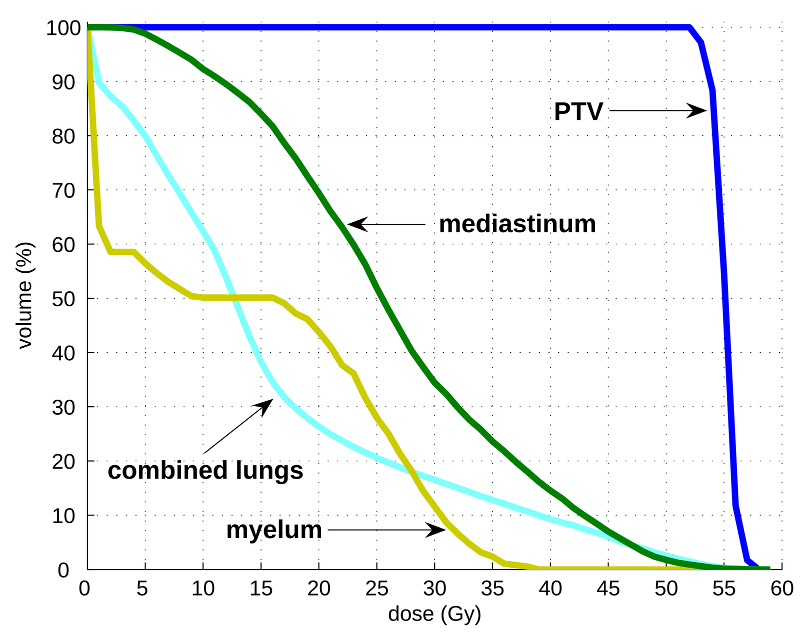
The algorithm has been tested with success on different cases: synthetic phantom, prostate, head-and-neck, and oesophagus. Here the case of the irradiation of an oesophagus tumor is presented. The plan used 5 equiangular beams. The optimization needed 45 iterations of the outer optimization to reach the dose objectives, and took about 7.5 minutes on a Intel Xeon $3.2\ GHz$ with Linux as operating system.

	constraints			realised dose statistics		
structure	$d^c\ (Gy)$	$V^c\ (\%)$	max (Gy)	$V\ (\%)$	max (Gy)	mean (Gy)
PTV	52.2	100.0	85.0	100.0	58.0	55.0
body	20.0	20.0	60.0	15.0	58.4	9.1
lungs	18.0	30.0	58.0	29.7	57.6	16.0
myelum	45.0	0.0	45.0	0.0	38.6	13.7
mediastinum	30.0	35.0	60.0	34.4	57.1	26.4

Dose objectives used in the calculation and the results. The prescribed dose in the PTV is 55 Gy.



This figure shows the calculated dose distribution in the slice central to the PTV.



This figure shows the DVH curves of the different structures.

Conclusions

With this work we have shown that it is possible to solve the problem of dose optimization with dose-volume constraints by using the mini-mization of a quadratic score function with prescribed doses in PTVs and zero dose levels in the organ at risk, combined with an automatic adaptation of voxel-based parameters. This approach asks for a fast solution of the minimization of the quadratic function. This has be done by using a new algorithm: BOXCQP, and by using calculation techniques for sparse matrices. Also the introduction of the smoothing term plays an important role in the minimization of the score function.

References

[1] Gustavson F G 1978 Two fast algorithms for sparse matrices: multiplication and permuted transposition *ACM Transactions on Mathematical Software* **4** 250-269

[2] Pissanetsky S 1984 *Sparse Matrix Technology* (London: Academic Press)

[3] Voglis C and Lagaris I E 2004 BOXCQP: An algorithm for bound constrained convex quadratic problems *1st IC-SCCE*